Pulse counting methods including multiplicity counting in nuclear safeguards

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Fluctuations in constant (low power) systems

- In a particle cascade with "branching" such as a fission chain, cosmic showers or family trees, the individual events are correlated. The branching (i.e. the particle multiplication) brings about correlations. Variance $\neq$ mean.

- Therefore, unlike in a Poisson process (= independent events) there is useful information in the fluctuations (i.e. in the "excess" variance over Poisson) beyond the information in the expectations, which can thus be utilized.

- However, as is known, family trees can also die out - also a statistical phenonenon. This is the start of the study of branching processes (The extinction of family trees, Watson and Galton).

Both of the above observations are relevant to the interest in neutron fluctuations in low power systems.

A historical interest

The extinction probability and the bomb

- Before dropping the first bomb on Hiroshima, Edward Teller suggested to explode it instead high above Tokyo during the night, as a deterring demonstration
- There were concerns that the bomb would not explode, among others due to the random character of the branching process (like the extinction of family trees)
- Feynman, Serber, de Hoffman, Fermi and others started investigating neutron fluctuations
- The original papers, which could confirm or deny their motivations, are still classified

Request for an LANL report in 2006

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Date:  Mon, 2 Oct 2006 17:19:51 +0200 (MEST)
From:  Imre Pazsit <imre@mail.nephy.chalmers.se>
Subject:  Inquiry about a report
To:  reports@lanl.gov

To whom it may concern

I would like to find a copy of an old wartime report which is
hard to identify.  It has the number LADC-250, it dates from
1944, and its authors are F. De Hoffman, E. Fermi and R. Feynman.
The order of the authors is not sure, neither that they all
appear on the report.  It is about the fluctuation of the number
of neutrons in a fission chain, generated in a multiplying
medium.

I appreciate very much if you can help us in finding the report.

Yours sincerely
Imre Pazsit
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The reply

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Date:  Mon, 02 Oct 2006 09:34:53 -0600 To:  Imre Pazsit
<imre@nephy.chalmers.se>
From:  Elaine Deschamp <edeschamp@lanl.gov>
Subject:  Re:  Inquiry about a report

We are sorry but due to a mandate from NNSA to the
laboratory and Research Library policies, we are unable to
provide technical reports until further notice.

Thank You,

Elaine
LANL Research Library
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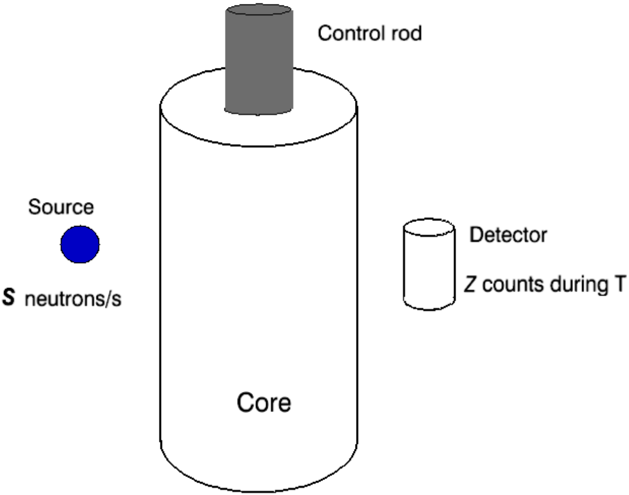
Neutron fluctuation based reactivity measurements



Determination of the multiplication factor/reactivity

Criticality is defined by the multiplication factor k , which is the fundamental eigenvalue of the transport equation:

$$k = N_i/N_{i-1}; \quad i = \text{generation number of neutrons in a chain}$$

Expectations of total number of neutrons generated in one chain:

$$1 + k + k^2 + \cdots = \frac{1}{1-k} \simeq \frac{1}{-\rho}$$

The flux level in a subcritical reactor with a source of intensity S, and hence the number of counts during a time T, will be proportional to $\dfrac{S}{-\rho}$

Determination of the multiplication factor/reactivity

Number of detector counts in a time interval T

$$\langle Z(T) \rangle = \frac{S}{-\rho} F(...) \cdot \varepsilon \cdot T$$

The function $F(...)$ is proportional to the leakage intensity of the neutrons, and the detector efficiency includes both material and geometry factors.

The source strength S, the exiting neutron flux F and the detector efficiency ε are all unknown.

Hence this method cannot be used without calibration to determine k or the reactivity

$$\rho = \frac{k-1}{k}$$

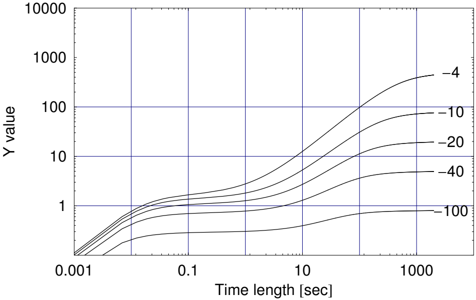
.

Use of fluctuations (second moment):

Feynman-alpha (variance to mean) method: dependence of the relative variance of the counts Z on the measurement time T where

$$\alpha = -\rho/\Lambda > 0 : \tag{1}$$

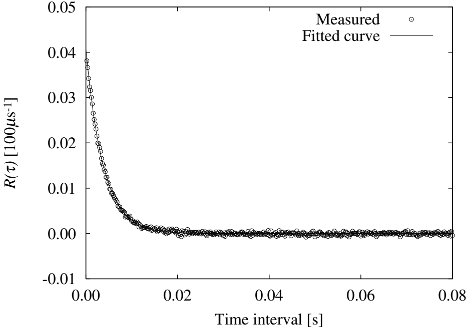
$$\boxed{\frac{\sigma_Z^2(T)}{\langle Z(T)\rangle} = 1 + Y(T) = 1 + \varepsilon A_1 \left(1 - \frac{1 - e^{-\alpha T}}{\alpha T}\right)} \tag{2}$$



Use of fluctuations (second moment):

Rossi-alpha method: dependence of the relative auto-covariance $R(\tau)$ of a detection at an arbitrary time followed by another one time τ later on the time lag τ and $\alpha = \rho/\Lambda$:

$$\boxed{R(\tau) = \frac{1}{2}\,\varepsilon\, A_1 \alpha\, e^{-\alpha\tau}} \tag{3}$$



The history of the theory

- Feynman, Serber and the others used phenomenological arguments to derive the Feynman-alpha formula (measurement of reactivity)
- In a similar way, the Italian cosmic ray physicist **Bruno Rossi** (1905 - 1943) suggested a formula for the correlations in detection of cosmic showers (the Rossi-alpha formula).
- The first step in developing the fundamental theory was taken by a Hungarian, **Lajos Jánossy** (1912 - 1978). He applied backward master equations for studying fluctuations in cosmic electron-photon showers ("regeneration point technique")
- The solid theoretical background of the Feynman- and Rossi-alpha formulas in reactor physics was given by **Lénárd Pál** (*Pál-Bell equations*) in a series of articles, the most known a paper in Nuovo Cimento Supplemento, in the late 1950's and early 1960's.

Theoretical framework: master equations

Master equation (Chapman-Kolmogorov equation) of Markovian processes for the transition probability $P(N, t| M, t_0)$ with $t > t' > t_0$:

$$P(N, t| M, t_0) = \sum_L P(N, t| L, t')P(L, t'| M, t_0) \tag{4}$$

Note: time (causality) from right to left here ($\leftarrow$).

A differential equation from (4) can be obtained in two ways: either by letting

$$t' \to t, \quad \text{i.e.} \quad t - t' = o(dt)$$

or

$$t' \to t_0, \quad \text{i.e.} \quad t' - t_0 = o(dt).$$

This will give two differential equations for the same quantity. They are called the **forward** and the **backward** Chapman-Kolmogorov or master equations, respectively.

Forward and backward equations

For getting solvable differential equations, we need to assume that the transition probabilities for infinitesimal times are known. For $N \neq M$, the transition probability tends to zero linearly with dt, thus one can write

$$P(N,t+dt|\,M,t) = P(N,t|\,M,t-dt) = W_{N,M}\,dt; \quad N \neq M \tag{5}$$

where the $W_{N,M}$ are called the *transition intensities*.

These are known from the physics of the process, as it will be illustrated soon.

The transition probabilites are not all independent. Since the probabilities have to sum up to unity, one has

$$W_{N,N}\,dt = 1 - \sum_{L \neq N} W_{L,N}\,dt \tag{6}$$

which is the probability of no event (transition) taking place during dt.

Forward and backward equations

Performing the limits $t' \to t$ and $t' \to t_0$, respectively, and using the transition probabilities, one obtains:

Forward equation:

$$\boxed{\frac{d}{dt}P(N,t|\,M,t_0) = \sum_{L\neq N} W_{N,L}P(L,t|\,M,t_0) - P(N,t|\,M,t_0)\sum_{L\neq N} W_{L,N}}$$

Backward equation:

$$\boxed{-\frac{d}{dt_0}P(N,t|\,M,t_0) = \sum_{L\neq M} P(N,t|\,L,t_0)\,W_{L,M} - P(N,t|\,M,t_0)\sum_{L\neq M} W_{L,M}}$$

One can see a similarity with the direct and adjoint neutron transport equations.

“Common sense” derivation for a branching process

Notations:

- Q: reaction intensity for *one neutron*:
- f_k: probability distribution of generating k neutrons in a reaction
- $q(z) = \sum_k f_k \, z^k$ the generating funtction of f_k (known).

Forward equation:

$$\boxed{p_n(t+dt) = p_n(t)(1 - n\,Q\,dt) \; + Q\,dt \sum_{k=0}^{n} f_k\,(n-k+1)p_{n-k+1}(t)}$$

Backward equation (“mixed” equation):

$$\boxed{p_n(t) = (1 - Q\,dt)p_n(t-dt) + Q\,dt \sum_{k=1}^{\infty} f_k \sum_{n_1+\cdots+n_k=n} \prod_{j=1}^{k} p_{n_j}(t)}$$

Forward and backward equations: for the generating function

Introducing the generating function $g(z,t)$ of the probability $p(n,t)$ of finding n particles at time t in a multiplying medium in the standard way, i.e.

$$g(z,t) = \sum_{n=0}^{\infty} p(n,t)\, z^n, \tag{7}$$

then the forward and backward equations have the following form:

Forward equation:

$$\boxed{\frac{\partial g(z,t)}{\partial t} = Q\,[q(z) - z]\;\frac{\partial g(z,t)}{\partial z}} \tag{8}$$

Backward equation:

$$\boxed{\frac{\partial g(z,t)}{\partial t} = Q\,\{q\,[g(z,t)] - g(z,t)\}} \tag{9}$$

The initial condition in both cases is $g(z,0) = z$.

The reason for the non-adjoint property

- As is well known, the transport operator (defining the "direct" transport equation) is not self-adjoint.
- Only operators which describe processes that are invariant to time reversal are self-adjoint. The transport process is not invariant to time reversal.
- One-group diffusion theory is invariant to time reversal, and hence it is self adjoint.
- In stochastic theory, the differences between the forward and the backward (adjoint) master equation, or their moments, are much larger.
- This is because the "violation" of time reversal is of a higher order for a branching process.

Time reversal of deterministic neutron transport

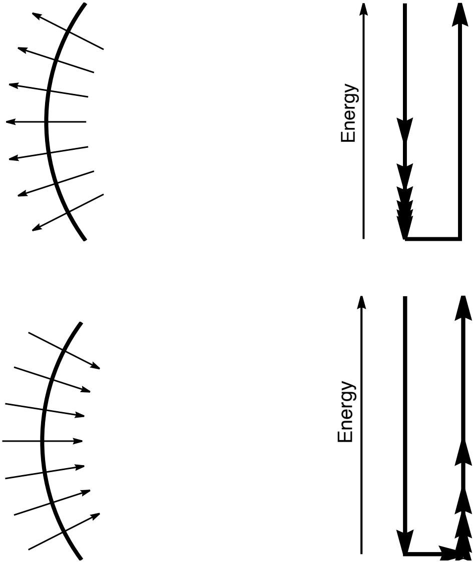
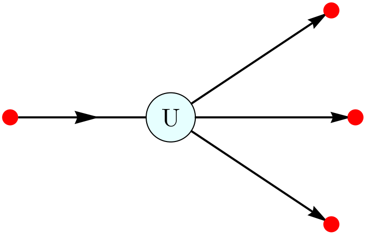
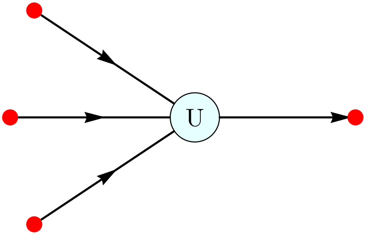


Illustration of time reversal in branching processes

Branching



“Inverse branching”



Feynman-alpha from a forward master equation

Following standard considerations, the forward master equation for $P(N,C,Z,t|t_0)$ can be written down by considering changes of the state of the system between t and $t+dt$, leading to

$$\frac{dP(N,C,Z,t|t_0)}{dt} = \lambda_c P(N+1,C,Z,t|t_0)(N+1) +$$

$$\lambda_d P(N+1,C,Z-1,t|,t_0)(N+1) +$$

$$\lambda_f \sum_n \sum_m P(N+1-n,C-m,Z,t|t_0)(N+1-n)p_f(n,m) +$$

$$S\ P(N-1,C,Z,t|t_0) + \lambda P(N-1,C+1,Z,t|t_0)(C+1) -$$

$$P(N,C,Z,t|t_0)[N(\lambda_f+\lambda_c+\lambda_d)+\lambda C + S]. \tag{10}$$

The initial condition associated with this equation reads as

$$P(N,C,Z,t=t_0|t_0) = \delta_{N,0}\delta_{C,0}\delta_{Z,0}. \tag{11}$$

Generating functions

By defining the generating functions

$$G(x,y,v,t|t_0) = \sum_N \sum_C \sum_Z x^N y^C v^Z P(N,C,Z,t|t_0) \tag{12}$$

and

$$g_f(x,y) = \sum_n \sum_m x^n y^m p_f(n,m), \tag{13}$$

the following equation is obtained from (10):

$$\boxed{\begin{aligned} \frac{\partial G(x,y,v,t|t_0)}{\partial t} &= \{\lambda_f[g_f(x,y) - x] - \lambda_c(x-1) - \lambda_d(x-v)\} \times \\ \frac{\partial G(x,y,v,t|t_0)}{\partial x} &+ \lambda(x-y)\frac{\partial G(x,y,v,t|t_0)}{\partial y} + (x-1)S\, G(x,y,v,t|t_0) \end{aligned}} \tag{14}$$

Notations for expectations

with the initial condition

$$G(x,y,v,t=t_0|t_0)=1; \qquad t_0 \le 0. \tag{15}$$

For the expectation of the random processes $\mathbf{N}(t)$, $\mathbf{C}(t)$ and $\mathbf{Z}(t,0)$, the notation of the expectation value is omitted, e.g.

$$\mathbf{E}\{\mathbf{N}(t)\} \equiv \langle \mathbf{N}(t) \rangle \equiv N(t) = \left. \frac{\partial G(x,y,v,t|t_0)}{\partial x} \right|_{x=y=v=1}. \tag{16}$$

Further moments

Further, one has

$$\left.\frac{\partial g_f(x,y)}{\partial x}\right|_{x=y=1} = \sum_n \sum_m n\ p_f(n,m) \equiv \langle \nu_p \rangle \equiv \langle \nu \rangle (1-\beta), \tag{17}$$

$$\left.\frac{\partial g_f(x,y)}{\partial y}\right|_{x=y=1} = \sum_n \sum_m m\ p_f(n,m) \equiv \langle \nu_d \rangle \equiv \langle \nu \rangle \beta, \tag{18}$$

The standard notations

$$\rho = \frac{\langle \nu \rangle \lambda_f - (\lambda_f + \lambda_c + \lambda_d)}{\langle \nu \rangle \lambda_f}, \tag{19}$$

$$\Lambda = \frac{1}{\langle \nu \rangle \lambda_f} \quad \text{and} \quad \varepsilon = \frac{\lambda_d}{\lambda_f} \tag{20}$$

are also introduced.

First moments

The three first moment equations read as follows:

$$\frac{dN(t|t_0)}{dt} = \frac{\rho - \beta}{\Lambda} N(t|t_0) + \lambda C(t|t_0) + S, \tag{21}$$

$$\frac{dC(t|t_0)}{dt} = \frac{\beta}{\Lambda} N(t|t_0) - \lambda C(t|t_0) \tag{22}$$

and

$$\frac{dZ(t,0|t_0)}{dt} = \lambda_d N(t,|t_0) = \varepsilon \lambda_f \; N(t|t_0), \qquad t \geq t_0. \tag{23}$$

The first two equations are actually the point kinetic equations (but for a subcritical system with a source), as seen in the previous Tutorial.

Second moments

For the sake of simplicity we introduce the modified second moment of the random variables $\mathbf{a}$ and $\mathbf{b}$ as follows:

$$\mu_{aa} \equiv \langle \mathbf{a}(\mathbf{a}-1)\rangle - \langle \mathbf{a}\rangle^2 = \sigma_a^2 - \langle \mathbf{a}\rangle \tag{24}$$

and

$$\mu_{ab} \equiv \langle \mathbf{ab}\rangle - \langle \mathbf{a}\rangle \langle \mathbf{b}\rangle \tag{25}$$

where $\mathbf{a}$ and $\mathbf{b}$ stand for either of the variables neutron population $\mathbf{N}$, precursor population $\mathbf{C}$ and detector count $\mathbf{Z}$.

Taking auto- and cross-derivatives, one obtains the following six equations:

Second moments

$$\frac{d\mu_{NN}(t|t_0)}{dt} = -2\alpha\mu_{NN}(t|t_0) + 2\lambda\mu_{NC}(t|t_0) + \lambda_f \left\langle \nu_p(\nu_p - 1)\right\rangle N(t|t_0), \quad (26)$$

$$\frac{d\mu_{NC}(t|t_0)}{dt} = -(\alpha+\lambda)\mu_{NC}(t|t_0) + \frac{\beta}{\Lambda}\mu_{NN}(t|t_0) + \lambda\mu_{CC}(t|t_0) + \lambda_f \left\langle \nu_p \, \nu_d \right\rangle N(t|t_0), \quad (27)$$

$$\frac{d\mu_{CC}(t|t_0)}{dt} = -2\lambda\mu_{CC}(t|t_0) + 2\frac{\beta}{\Lambda}\mu_{NC}(t|t_0) + \lambda_f \left\langle \nu_d(\nu_d - 1)\right\rangle N(t|t_0), \quad (28)$$

$$\frac{d\mu_{NZ}(t,0|t_0)}{dt} = -\alpha\mu_{NZ}(t,0|t_0) + \lambda\mu_{CZ}(t,0|t_0) + \varepsilon\lambda_f\mu_{NN}(t|t_0), \quad (29)$$

$$\frac{d\mu_{CZ}(t,0|t_0)}{dt} = -\lambda\mu_{CZ}(t,0|t_0) + \frac{\beta}{\Lambda}\mu_{NZ}(t,0|t_0) + \varepsilon\lambda_f\mu_{NC}(t|t_0), \quad (30)$$

$$\frac{d\mu_{ZZ}(t,0|t_0)}{dt} = 2\varepsilon\lambda_f\mu_{NZ}(t,0|t_0). \quad (31)$$

The need for nuclear safeguards

- Nuclear materials are widespread: industry, medicine, electricity production, agriculture etc.
- Most materials are well known, but still require regular control and verification.
- If illicit trafficking of nuclear material is not prevented the possible results are:
  - Proliferation
  - Health risks during transport and human contact.
  - Material could be used in radioactive dispersion devices - “Dirty bombs”
- Sources that are not accounted for (Orphan sources) are increasing in number and hard to quantify and detect.

Application in safeguards: multiplicity counting

Traditional multiplicity counting

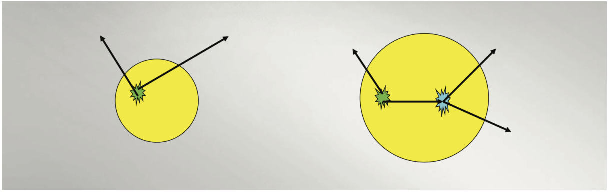
- Multiplicity counting is a method to determine the unknown parameters, primarily the fissile mass, of an unknown item by measuring the count rate of spontaneously emitted radiation (neutrons).
- It is performed by counting discrete detection events, i.e. using neutron detectors in pulse mode.
- Because the count rate depends on three unknown parameters of the sample (see later), we need three independent measured quantities.
- These are the singles, doubles and triples count rates ($\boldsymbol{S}$, $\boldsymbol{D}$ and $\boldsymbol{T}$ rates), i.e. the frequency of measuring one, two or three neutrons in coincidence ("simultaneously").
- By deriving analytical relationships between the three unknown parameters and the $\boldsymbol{S}, \boldsymbol{D}$ and $\boldsymbol{T}$ rates, the parameters can be unfolded from the measured rates.

Internal multiplication

Neutron emission by spontaneous fission from a sample: in multiplets (compound Poisson distribution in time).

For larger samples, internal multiplication will occur (by induced fission), considered as instantaneous.

Some source events are not multiplets: (α,n) reactions **not determined exclusively by the fissile mass**.



Unknowns:

- the sample mass (fission rate $\boldsymbol{F}$),
- the fraction of (α,n) neutrons in the total source neutrons ($\boldsymbol{\alpha}$),
- the internal multiplication $\boldsymbol{M}$.

Methodology

These three unknowns are determined from the $\boldsymbol{S}, \boldsymbol{D}$ and $\boldsymbol{T}$ rates.

Because only a backward approach can be used, derivation of the relationship between the $\boldsymbol{S}, \boldsymbol{D}$ and $\boldsymbol{T}$ rates and the unknown parameters $\boldsymbol{F}$, $\boldsymbol{\alpha}$ and $\boldsymbol{M}$ goes in two steps.

- First, one derives the first three factorial moments ("Böhnel moments") of the number of neutrons emitted from the sample by a source event from a **time-independent** backward type master equation for discrete events, assuming a first collision probability p (which determines the internal multiplication);
- Then, by accounting to the intensity of the source events (through the fission rate $\boldsymbol{F}$, these moments are converted into $\boldsymbol{S}, \boldsymbol{D}$ and $\boldsymbol{T}$ rates (detections per second).

First step: derivation of the factorial moments

Master equations for the number distributions of neutrons emitted from the sample.

$p(n)$ - number of neutrons emitted from the item due to *one neutron*;

$P(n)$ - number of neutrons emitted from the item due to a *source event*;

$h(z)$ and $H(z)$ - their generating functions.

$$p(n) = (1-\mathrm{p})\delta_{n,1} + \mathrm{p} \sum_{k=1}^{\infty} p_r(k) \prod_{\substack{i=1 \\ \{n_1+n_2+\ldots+n_k=n\}}}^{k} p(n_i) \tag{32}$$

$$P(n) = \sum_{k=1}^{\infty} p_s(k) \prod_{\substack{i=1 \\ \{n_1+n_2+\ldots+n_k=n\}}}^{k} p(n_i) \tag{33}$$

$p_r(k)$ and $p_s(k)$ are the number distributions in an induced fission and the source neutrons per emission event, respectively.

Derivation of the factorial moments (cont)

The generating functions are defined as

$$h(z) = \sum_{n=0}^{\infty} p(n)\, z^n; \quad H(z) = \sum_{n=0}^{\infty} P(n)\, z^n \tag{34}$$

For the generating functions one obtains from (32) and (33)

$$\boxed{\begin{aligned} h(z) &= (1 - \mathrm{p})\, z + \mathrm{p} q_r\, [h(z)] &(35)\\ H(z) &= q_s\, [h(z)] &(36)\end{aligned}}$$

From these, the factorial moments $\widetilde{\nu}_i$, $i = 1{,}2{,}3$ of the number of neutrons emitted from the sample by a source event are given as

$$\widetilde{\nu}_i = \left. \frac{d^i H(z)}{d\, z^i} \right|_{z=1} \tag{37}$$

Factorial moments

- The equations for the generating functions are higly non-linear, and cannot be solved explicitly.
- However, the factorial moments and the values of $P(n)$ can be obtained analytically in a recursive manner from linear equations.

Neutron Factorial Moments

- As previously mentioned the factorial moments are routinely used in safeguard measurements since they are related to measurable quantities.
- The first three moments are normally used: singles, doubles, triples.
- Factorial moments are calculated as derivatives of the generating functions evaluated at $z = 1$.

First moment

First moments (singles)

$$\widetilde{\nu}_1 = \nu_{s,1}\; h_1, \tag{38}$$

where

$$h_1 = \left[\frac{dh(z)}{dz}\right]_{z=1} \equiv \nu_1.$$

It follows from (35) that $h_1 = 1 - \mathrm{p} + \mathrm{p}\,\nu_{r,1}\; h_1$. from which one obtains

$$h_1 = \frac{1-\mathrm{p}}{1-\mathrm{p}\,\nu_{r,1}} \equiv \mathbf{M}, \qquad \text{where} \qquad \mathrm{p}\,\nu_{r,1} < 1. \tag{39}$$

The expression $\mathbf{M} \equiv h_1$ is called **leakage multiplication**.

The leakage multiplication is not known, because the first collision probability p is not known. The unknown parameter p will not appear in the continuation, only $\mathbf{M}$.

Doubles, triples

Second moments (doubles)

In a similar way one obtains

$$\widetilde{\nu}_2 = \mathbf{M}^2 \left\{ \nu_{s,2} + \frac{\mathrm{p}}{1 - \mathrm{p}\, \nu_{r,1}} \nu_{s,1}\, \nu_{r,2} \right\} = \mathbf{M}^2 \left\{ \nu_{s,2} + \frac{\mathbf{M} - 1}{\nu_{r,1} - 1} \nu_{s,1}\, \nu_{r,2} \right\}. \tag{40}$$

Third moments (triples)

$$\widetilde{\nu}_3 = \mathbf{M}^3 \left\{ \nu_{s,3} + \frac{\mathbf{M} - 1}{\nu_{r,1} - 1} \left(3\, \nu_{s,2}\, \nu_{r,2} + \nu_{s,1}\, \nu_{r,3} \right) + 3 \left(\frac{\mathbf{M} - 1}{\nu_{r,1} - 1} \right)^2 \nu_{s,1}\, \nu_{r,2}^2 \right\}. \tag{41}$$

Symbolic computation (with the use of Mathematica™) allows calculation of very high order moments.

Second step: conversion to detection rates

Denote the intensity of the source events $Q = F\,(1 + \alpha\,\nu_{sf,1})$

Then the detection intensity of the multiplets $\boldsymbol{S}, \boldsymbol{D}$ and $\boldsymbol{T}$ rates is given as

$$\boldsymbol{S} = \varepsilon\, Q \left\langle \binom{n}{1} \right\rangle = \varepsilon\, Q\, \frac{\widetilde{\nu}_1}{1!} \tag{42}$$

$$\boldsymbol{D} = \varepsilon^2\, Q \left\langle \binom{n}{2} \right\rangle = \varepsilon^2\, Q\, \frac{\widetilde{\nu}_2}{2!} \tag{43}$$

$$\boldsymbol{T} = \varepsilon^3\, Q \left\langle \binom{n}{3} \right\rangle = \varepsilon^3\, Q\, \frac{\widetilde{\nu}_3}{3!} \tag{44}$$

where ε is the detection efficiency.

From this and the formulae derived so far, the detection *rates* of the singles, doubles and triples can be easily calculated.

Detection rates

The final results for the detection rates of singles, doubles and triples are given as

$$\boldsymbol{S} = \boldsymbol{F}\varepsilon\,\mathbf{M}\,\nu_{sf,1}(1+\boldsymbol{\alpha}) \tag{45}$$

$$\boldsymbol{D} = \frac{\boldsymbol{F}\varepsilon^2 f_d\,\mathbf{M}^2}{2}\left[\nu_{sf,2} + \left(\frac{\mathbf{M}-1}{\nu_{i1}-1}\right)\nu_{sf,1}(1+\boldsymbol{\alpha})\nu_{i2}\right] \tag{46}$$

$$\begin{aligned}\boldsymbol{T} = \frac{\boldsymbol{F}\varepsilon^3 f_t\,\mathbf{M}^3}{6}\Bigg[\nu_{sf,3} + \left(\frac{\mathbf{M}-1}{\nu_{i1}-1}\right)\;[3\nu_{sf,2}\nu_{i2} + \nu_{sf,1}(1+\boldsymbol{\alpha})\nu_{i3}] \\ + 3\left(\frac{\mathbf{M}-1}{\nu_{i1}-1}\right)^2 \nu_{sf,1}(1+\boldsymbol{\alpha})\nu_{i2}^2\Bigg]\end{aligned} \tag{47}$$

where

$$\boxed{\mathbf{M} = \frac{1-\mathrm{p}}{1-\mathrm{p}\,\nu_{r,1}}} \tag{48}$$

is the leakage multiplication, related to p, the uniform first collision probability.

The inversion procedure

The unknown sample parameters $\mathbf{M}$, F and α can be determined from the measured values of S, D and T by inverting the above expressions.

One obtains a third order equation for the leakage multiplication $\mathbf{M}$:

$$a + b\mathbf{M} + c\mathbf{M}^2 + \mathbf{M}^3 = 0 \tag{49}$$

$$a = \frac{-6\,T\nu_{sf,2}(\nu_{i1}-1)}{\varepsilon^2 f_t S\left[\nu_{sf,2}\nu_{i3} - \nu_{sf,3}\nu_{i2}\right]} \tag{50}$$

$$b = \frac{2D\left[\nu_{sf,3}(\nu_{i1}-1) - 3\nu_{i2}\nu_{sf,2}\right]}{\varepsilon f_d S\left[\nu_{sf,2}\nu_{i3} - \nu_{sf,3}\nu_{i2}\right]} \tag{51}$$

$$c = \frac{6\nu_{i2}\nu_{sf,2}D}{\varepsilon f_d S\left[\nu_{sf,2}\nu_{i3} - \nu_{sf,3}\nu_{i2}\right]} - 1 \tag{52}$$

Outlook, current activity

Current research activity includes:

- Extending the "point model" of multiplicity counting to space-angle-energy dependent cases
- Using continuous signals of fission chambers for multiplicity counting and reactivity determination
- Start-up with a weak source (UK)
- Stochastic modelling of the detection of scintillation detectors with Geant4

In summary, neutron fluctuations in low power systems is a very active area of current research.