Neutron fluctuations in safeguards - some current developments

Imre Pázsit

Chalmers University of Technology
Division of Subatomic, High Energy and Plasma Physics
Nuclear Engineering Group

IMORN-31
Delft, 2024-09-10

Objectives

- The singles, doubles and triples rates of detected neutrons, used in non-destructive assay, are based on the first three factorial moments of the number of particles emitted from a fissile item per one source event.
- These factorial moments ("Böhnel moments") were traditionally derived in the so-called point model, in which the space, angular and energy dependence of the neutron transport is neglected.
- In present work, we extended the calculation of the factorial moments to include space, angular and energy dependence, as well as to include scattering.

Extension to a one-speed transport model

- To start with, spherical geometry will be considered, and (α, n) reactions neglected
- Distances are expressed in units of the mean free path
- The number distribution $p_f(n)$ of neutrons from an induced fission (subscript f) and from a source event $p_s(n)$ (subscript s) as well as their generating functions will be denoted as

$$p_f(n); \quad q_f(z) = \sum_{n=0}^{\infty} p_f(n)\, z^n; \quad \text{and} \quad p_s(n); \quad q_s(z) = \sum_{n=0}^{\infty} p_s(n)\, z^n.$$

These are known quantities.

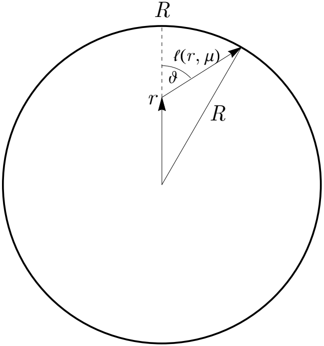
In the calculations, we will need the first three factorial moments $\nu_{f,i}$ and $\nu_{s,i}$, $i = 1,2,3$ of these distributions, i.e.

$$\nu_{f,i} = \sum_{n=0}^{\infty} n\,(n-1)...(n-i+1)\, p_f(n) = \left.\frac{\mathrm{d}^i q_f(z)}{\mathrm{d}z^i}\right|_{z=1} \tag{1}$$

Equations for a sphere

Variables in spherical geometry:

- the radial position r of the starting neutron
- the cosine μ of the polar angle ϑ between the position vector and the velocity vector of the neutron
- $\ell(r, \mu)$ is the distance to the boundary from the point r along μ.



Equations for a sphere: single starting neutron

The equation for the probability distribution $p(n\,|\,r,\mu)$ that a source neutron with coordinates $(r,\,\mu)$ will lead to the emission of n neutrons reads as

$$p(n|\,r{,}\mu) = e^{-\ell(r,\,\mu)}\,\delta_{n,1}+$$

$$\int_0^{\ell(r,\,\mu)} ds\; e^{-s}\;\sum_0^{\infty} p_r(k) \sum_{n_1+n_2+\cdots+n_k=n} \int_{-1}^{1} \frac{d\mu_1}{2}\frac{d\mu_2}{2}\ldots\frac{d\mu_k}{2}\times \tag{2}$$

$$p(n_1|\,r'(s),\mu_1)\,p(n_2|\,r'(s),\,\mu_2)\ldots p(n_k|\,r'(s){,}\mu_k).$$

Here,

$$r' \equiv r'(s) = |\boldsymbol{r}'| \equiv |\boldsymbol{r}+s\boldsymbol{\Omega}] = \sqrt{r^2+s^2+2\,r\,s\,\mu}, \tag{3}$$

μ_k is the cosine between $\boldsymbol{r}'$ and $\boldsymbol{\Omega}'_k$, and $\ell(r{,}\mu)$ is given as

$$\ell(r{,}\mu) = -r\mu+\sqrt{(r\,\mu)^2+(R^2-r^2)} = -r\mu+\sqrt{r^2\,(\mu^2-1)+R^2}. \tag{4}$$

Equations for the generating function

Introducing the generating function $g(z|\,r,\,\mu)$ of $p(n|\,r,\,\mu)$ in the usual way, i.e.

$$g(z|\,r,\,\mu) = \sum_{n=0}^{\infty} z^n \, p(n|\,r,\,\mu), \tag{5}$$

one obtains the more compact equation

$$g(z|\,r,\mu) = z\,e^{-\ell(r,\mu)} + \int_0^{\ell(r,\mu)} \mathrm{d}s\, e^{-s}\, q_r\,[\; g(z|\,r'(s))] \tag{6}$$

where $g(z\,|\,r)$ is the "scalar" (angularly integrated) generating function and $q_r[...]$ is the generating function of $p_r(k)$, with the quantity in the square brackets being its argument.

The scalar generation function by one source neutron

Introducing the scalar (angularly integrated) probability distribution as

$$p(n|\, r) = \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu \, p(n|\, r,\mu), \tag{7}$$

and its generating function

$$g(z|\, r) = \sum_{n=0}^{\infty} z^n \, p(n|\, r), \tag{8}$$

we arrive to the substantially simpler equation for the generating function:

$$g(z|\, r) = z\, g_0(z|\, r) + \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu \int_{0}^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \, q_r \left[g(z|\, r'(s)) \right] \tag{9}$$

where

$$g_0(z|\, r) = \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu \, e^{-\ell(r,\mu)}. \tag{10}$$

and

$$q_r(z) = \sum^{\infty} z^k \, p_r(k).$$

The source event induced distribution

The expression for the generating function $G(z)$ of the source event induced distribution reads as

$$G(z) = \frac{1}{V} \int_V \mathrm{d}\boldsymbol{r}\, q_s \left[\, g(z|\, r)\right] = \frac{3}{R^3} \int_0^R r^2\, q_s \left[g(z|\, r)\right]\, \mathrm{d}r \tag{11}$$

- is not affected by the inclusion of new reaction types;
- not an equation to be solved, only a recipe how to calculate $G(z)$, or its moments from the scalar single-neutron induced generating function $g(z \mid r)$.

Factorial moments $n, N; m, M$ and w, W

From (9) and (11), equations can be derived for the moments by derivation.

First moments:

$$\frac{\partial}{\partial z} g(z|\, r)\,|_{z=1} = \langle n(r) \rangle \equiv n(r), \tag{12}$$

and

$$\frac{\partial}{\partial z} G(z)\,|_{z=1} = \langle N \rangle \equiv N = \nu_1. \tag{13}$$

Second moments:

$$\frac{\partial^2}{\partial z^2} g(z|\, r)\,|_{z=1} = \langle n(r)\,(n(r) - 1) \rangle \equiv m(r) \tag{14}$$

and

$$\langle N\,(N-1) \rangle \equiv M = \nu_2 \tag{15}$$

Third moments

$$\frac{\partial^3}{\partial z^3} g(z|\, r)\,|_{z=1} = \langle n(r)\,(n(r) - 1)(n(r) - 2) \rangle \equiv w(r) \tag{16}$$

and

$$\langle N\,(N-1)\,(N-2) \rangle \equiv W = \nu_3. \tag{17}$$

Results: single neutron induced moments

From (9), the following equations can be derived for the first three factorial moments of the number of particles leaving the sample:

First moment
$$\boxed{n(r,\mu) = e^{-\ell(r,\mu)} + \nu_{r,1} \int_0^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \, n(r'(s))} \tag{18}$$

Second moment
$$\boxed{m(r,\mu) = A(r,\mu) + \nu_{r,1} \int_0^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \, m(r'(s))} \tag{19}$$

with
$$A(r,\mu) = \nu_{r,2} \int_0^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \, n^2(r'(s)). \tag{20}$$

Third moment
$$\boxed{w(r,\mu) = B(r,\mu) + \nu_{r,1} \int_0^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \, w(r'(s))} \tag{21}$$

with
$$B(r,\mu) = \int_0^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \left\{ \nu_{r,3} \, n^3(r'(s)) + 3 \, \nu_{r,2} \, n(r'(s)) \, m(r'(s)) \right\} \tag{22}$$

Source event induced moments

Similarly, from (11), expressions can be derived for the source event induced moments. These show that, in possession of the **scalar** single neutron induced moments $n(r)$, $m(r)$ and $w(r)$, these can be immediately calculated.

First moment:

$$\boxed{N = \frac{3\,\nu_{s,1}}{R^3}\int_0^R r^2\, n(r)\,\mathrm{d}r} \tag{23}$$

Second moment

$$\boxed{M = \frac{3}{R^3}\int_0^R r^2\left\{\nu_{s,2}\, n^2(r) + \nu_{s,1}\, m(r)\right\}\,\mathrm{d}r} \tag{24}$$

Third moment

$$\boxed{W = \frac{3}{R^3}\int_0^R r^2\left\{\nu_{s,3}\, n^3(r) + 3\,\nu_{s,2}\, n(r)\, m(r) + \nu_{s,1}\, w(r)\right\}\,\mathrm{d}r} \tag{25}$$

Hence, we only need the scalar moments $n(r)$, $m(r)$ and $w(r)$ in the continuation.

Solution with a collision number expansion

The starting (zeroth) term consists of the expectation of the scalar non-collided part

$$n_0(r) = \frac{1}{2} \int_{-1}^{1} e^{-\ell(r,\mu)} \, \mathrm{d}\mu.$$

The scalar once-collided part $n_1(r)$ is then given as

$$n_1(r) = \frac{\nu_{f,1}}{2} \int_{-1}^{1} \mathrm{d}\mu \int_0^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \, n_0(r'(s)) \tag{26}$$

and in general, the kth term of the expansion is given as

$$n_k(r) = \frac{\nu_{f,1}}{2} \int_{-1}^{1} \mathrm{d}\mu \int_0^{\ell(r,\mu)} \mathrm{d}s \, e^{-s} \, n_{k-1}(r'(s)). \tag{27}$$

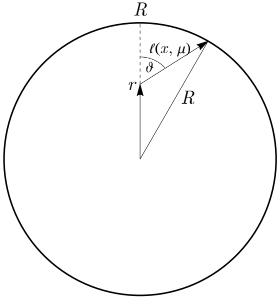
The full solution for the scalar first moment is obtained by summing up to all collision numbers, i.e.

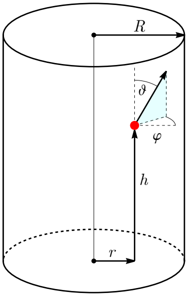
$$n(r) = \sum_{k=0}^{\infty} n_k(r). \tag{28}$$

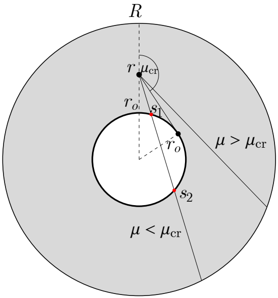
Results with the space-dependent model

- Calculations were made for spheres, cylinders and shells with a central cavity
- A comparison with the point model showed that the space-dependent model predicts higher factorial moments.
- More important, due to the above, the point model underestimates the fission rate (and hence the mass of the fissionable isotope, i.e. that of the ^{240}Pu).

The geometries considered







Results with the space-dependent model

- Calculations were made for spheres, cylinders and shells with a central cavity
- A comparison with the point model showed that the space-dependent model predicts higher factorial moments.
- More important, due to the above, the point model underestimates the fission rate (and hence the mass of the fissionable isotope, i.e. that of the ^{240}Pu).

Results with the space-dependent model

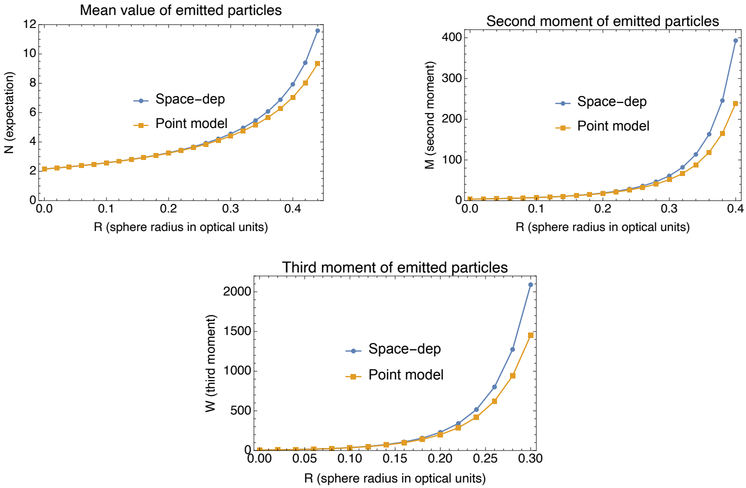
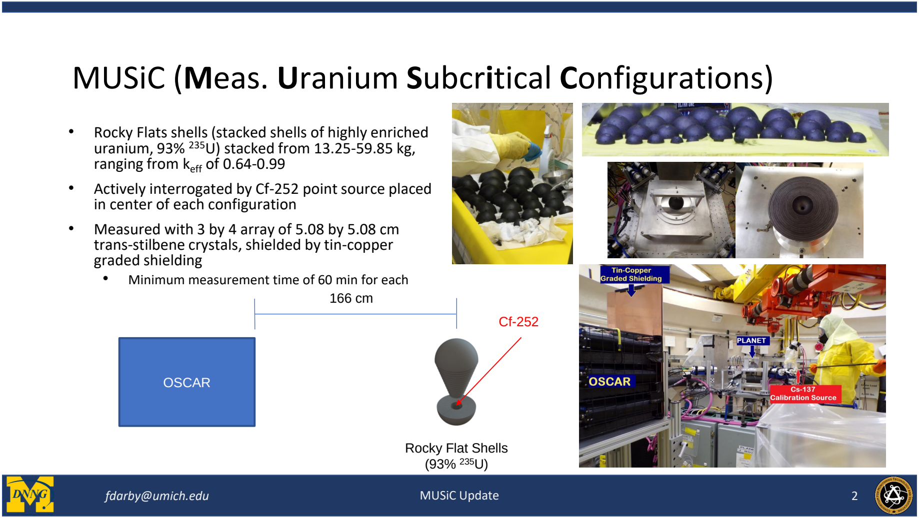


Figure 1: The first three factorial moments as functions of the radius of a sphere containing 80% ^{239}Pu and 20%^{240}Pu, as calculated by the point model and the space-dependent model

Results with the space-dependent model

- However, comparison with measurements made on the Rocky Flats Shells in the MUSIC experimental program (collaboration between LANL and the University of Michigan), the theoretical results predicted (much) lower factorial moments than the measurements
- The reason for the difference between measured and calculated values was traced down to the neglection of scattering reactions
- In a one-speed theory, elastic scattering can be incorporated with some approximations.



Reference: LA-UR-21-31641

Accounting for elastic scattering in a one-speed model

Assumptions:

- the energy loss and anisotropy of the neutrons scattered elastically on heavy nuclei is neglected
- then, scattering can be treated as a fission event resulting in one neutron
- can be handled analogously as the (α,n) neutrons in the source distribution
- define the number distribution of neutrons $p_r(k)$ *per reaction* as

$$p_r(k) = c_f \, p_f(k) + c_{el} \, \delta_{k,1} \tag{29}$$

where

$$c_f \equiv \frac{\Sigma_f}{\Sigma_T}; \quad c_{el} \equiv \frac{\Sigma_{el}}{\Sigma_T} \quad \text{and} \quad \Sigma_T = \Sigma_f + \Sigma_{el}. \tag{30}$$

This means that the moments $\nu_{f,i}$, $i = 1,2,3$ of induced fission have to be replaced by the moments $\nu_{r,i}$ of the number of secondaries per reaction:

$$\nu_{r,i} = c_f \, \nu_{f,i} + c_{el} \, \delta_{k,1} \tag{31}$$

Correspondingly, the optical path has to be scaled by the total cross section, instead of the fission cross section as in the preceding work.

Results

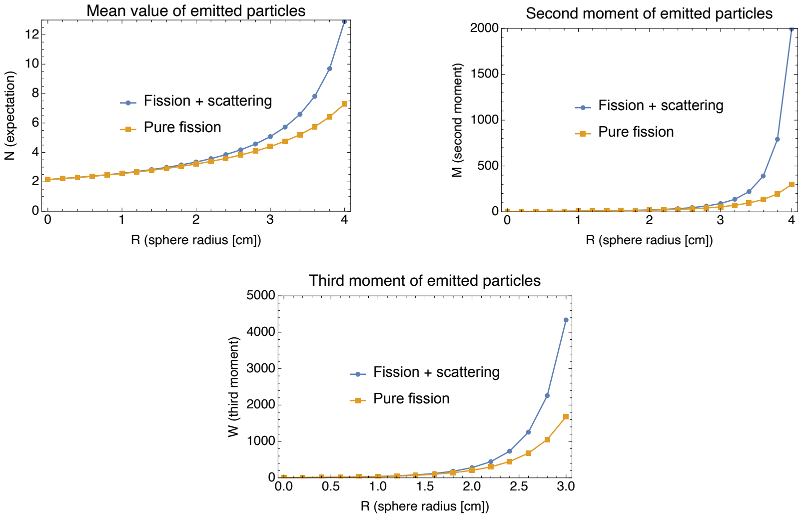


Figure 2: The first three factorial moments as functions of the radius of a pure ^{239}U sphere, with and without elastic scattering being included.

Discussion

- Accounting for elastic scattering, the factorial moments increased, showing the importance of including scattering
- however, the calculated values increased too much, so there was still no significant improvement in the agreement with measurements
- the remaining reason is the lack of accounting for inelastic scattering. It decreases the energy of neutrons to regions where the fission neutron multiplicities are lower than at the average source energy
- heuristic recipe: use the cross sections and the fission neutron multiplicities at 1 MeV instead of the 2 MeV of the average source energy
- with this fix, good agreement with the measurements were obtained (next slide)
- however, instead of a fix, a proper treatment of the inelastic scattering needs to be included $\rightarrow$ extension to energy dependent transport calculations.

Comparison between calculations and measurements

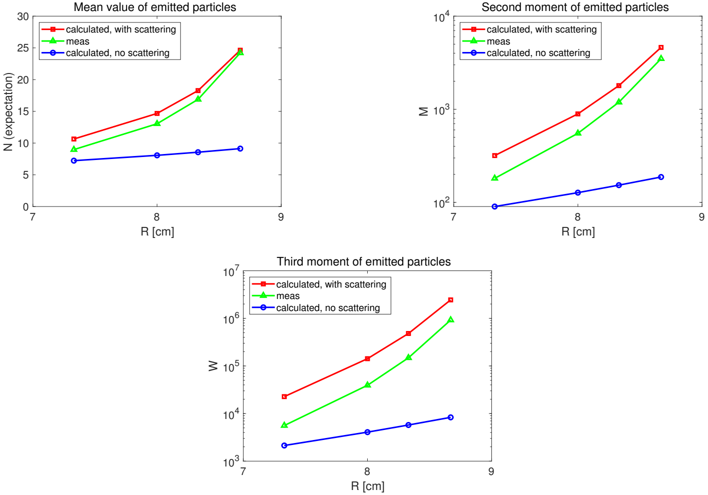


Figure 3: Comparison of measured and calculated first, second and third moments of the number of neutrons emitted from the Rocky Flats Shells (93.5% enriched ^{235}U) for four different outer radii at the neutron energy of 1 MeV

Elastic and inelastic scattering: energy-dependent transport theory

- (Elastic) scattering becomes anisotropic
- but the generating function $g(z|\, r,\, \mu,\, E)$, and hence also its moments, still depend only on μ what regards the angular variable, and do not depend on the azimuthal angle φ.

We need to introduce the energy dependent cross sections

$$\Sigma_f(E), \qquad \Sigma_{el}(E) \quad \text{and} \quad \Sigma_{in}(E) \tag{32}$$

and the total cross section $\Sigma_T(E) \equiv \Sigma(E)$ as

$$\Sigma(E) = \Sigma_f(E) + \Sigma_{el}(E) + \Sigma_{in}(E) \tag{33}$$

In addition, the fission number distribution $p_f(n,E)$, as well as its generating function $q_f(n,E)$ become dependent on the energy of the fissioning neutron.

This means that the factorial moments of neutrons per fission, $\nu_{f,i} = \nu_{f,i}(E)$ will also be energy dependent. This is the main reason of the energy dependence of the factorial moments of the neutrons emitted from the item.

Scattering functions

Further, the scattering functions (densities) for both the elastic and inelastic scattering, as well as for the fission process, are needed:

$$f_{el}(\mu \to \mu', E \to E'), \qquad f_{in}(\mu \to \mu', E \to E') \quad \text{and} \quad \chi(E \to E') \tag{34}$$

Unlike in ordinary transport theory, the elastic, inelastic and fission scattering functions cannot be combined into one common scattering function with a norm > 1, because they have different number distributions.

The energy and angular dependence of these two scattering functions will be discussed in the following.

Scattering functions

Elastic scattering

Elastic scattering is not isotropic in the lab system, and there is a direct relationship between energy loss and the scattering angle:

$$f_{el}(\mu \to \mu', E \to E') \equiv f_{el}(\mu_0, E \to E') = \frac{\Delta(E' - \alpha\, E)}{(1-\alpha)\, E}\, \delta\left(\mu_0 - S(E,E')\right); \qquad E' \leq E \tag{35}$$

with

$$\alpha = \left[\frac{A-1}{A+1}\right]^2, \quad S = \frac{1}{2}\left[(A+1)\sqrt{\frac{E'}{E}} - (A-1)\sqrt{\frac{E}{E'}}\right], \tag{36}$$

Here $\Delta(x) =$ the unit step function, and $\mu_0 = \mathbf{\Omega} \cdot \mathbf{\Omega}'$ is the cosine of the scattering angle. In the transport equation this term will therefore enter in the form

$$f_{el}(\mathbf{\Omega} \cdot \mathbf{\Omega}', E \to E') = \frac{\Delta(E' - \alpha\, E)}{2\,\pi\,(1-\alpha)\, E}\, \delta\left(\mathbf{\Omega} \cdot \mathbf{\Omega}' - S(E,E')\right); \qquad E' \leq E \tag{37}$$

and will have to be integrated w.r.t. $\mathbf{\Omega}'$.

Inelastic scattering

Inelastic scattering

- It consists of both discrete level and a continuum scattering
- Inelastic scattering from heavy nuclei can be regarded isotropic in the laboratory system.

Hence, one has

$$f_{in}(\mu_0, E \to E') = \frac{1}{2} f_{in}(E \to E') = \frac{f_{in,d}(E \to E')}{2} + \frac{f_{in,c}(E \to E')}{2}; \qquad E' \leq E \tag{38}$$

Discrete level scattering

From kinematic considerations, the energy loss function for inelastic scattering with excitation to energy level Q_k is given as

$$f_{in}(E \to E' \,|\, Q_k) = \frac{1}{E\,(1-\alpha)\,\sqrt{1-\varepsilon_k/E}} \tag{39}$$

for

$$\left(\frac{A\,\sqrt{1-\varepsilon_k/E} - 1}{A+1}\right)^2 E \leq E' \leq \left(\frac{A\,\sqrt{1-\varepsilon_k/E} + 1}{A+1}\right)^2 \tag{40}$$

and 0 otherwise.

Inelastic scattering (discrete levels)

In the above,

$$\varepsilon_k = \frac{A+1}{A}\, Q_k \tag{41}$$

Thus for discrete level scattering, the inelastic scattering function is given by summing up for all excitation levels:

$$f_{in,d}(E \to E') = \sum_k p_k(E) f_{in}(E \to E' \,|\, Q_k) \tag{42}$$

where

$$p_k(E) = \frac{\sigma_k(E)}{\sum_i \sigma_i(E)} \tag{43}$$

is the conditional probability that the inelastic scattering with neutron energy E will excite the energy level Q_k of the nucleus.

Inelastic scattering (continuum)

Continuum scattering

- the levels are assumed to be continuously distributed from a certain threshold energy Q_0 with level density $w(Q)$;
- The neutron cross section $\sigma_k(E)$ for scattering with excitation of level k is replaced by $\sigma(E,Q)$, i.e. with the cross section for scattering with excitation of a level with energy Q.
- the probability p_k is replaced by $p(E,Q)$, by including the level density

$$p(E,Q) = \frac{\sigma(E,Q)\,w(Q)}{\int \sigma(E,Q)\,w(Q)\,\mathrm{d}Q} \tag{44}$$

Hence, we finally have

$$f_{in,c}(E \to E') = \int p(E,Q) f_{in}(E \to E' |\, Q)\,\mathrm{d}Q. \tag{45}$$

For fixed E' (and E) the range of Q can similarly be determined as in the discrete case, but it will not be described here.

Energy dependent master equation

Because the elastic scattering is anisotropic, one cannot write down an equation directly for the (energy dependent) scalar generating function and its moments, one has to keep the angular dependence.

Generating function:

$$g(z|\,r,\,\mu,\,E) = z\,e^{-\ell(r,\mu)\Sigma(E)} + \frac{\Sigma_{el}(E)}{2\pi(1-\alpha)\,E}\,times$$

$$displaystyle \int_0^{\ell(r,\mu)} \mathrm{d}s\,e^{-s\,\Sigma(E)} \int_{\alpha E}^{E} \mathrm{d}E' \int_{-1}^{1} \mathrm{d}\mu' \int_0^{2\pi} \mathrm{d}\varphi'\,\delta\left(\mathbf{\Omega}\cdot\mathbf{\Omega}' - S(E,E')\right)\,g(z|\,r'($$

$$\Sigma_{in}(E)\int_0^{\ell(r,\mu)} \mathrm{d}s\,e^{-s\,\Sigma(E)} \int_0^{E} \mathrm{d}E'\,f_{in}(E\rightarrow E')\,g(z|\,r'(s),\,E') +$$

$$\Sigma_f(E)\int_0^{\ell(r,\mu)} \mathrm{d}s\,e^{-s\,\Sigma(E)}\,q_f\left[\int_0^{E_{max}} \mathrm{d}E'\,\chi(E\rightarrow E')\,g(z|\,r'(s),E'),\,E'\right] \tag{46}$$

where

$$g(z|\,r,\,E) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu'\,g(z|\,r,\,\mu\,E) \tag{47}$$

is the “scalar” (angularly integrated) generating function of the leakage importance.

Treatment of the elastic scattering

Due to the Dirac-delta function in the elastic scattering term, one should in principle be able to calculate one integral analytically. However, this calculations is by far not trivial.

First, we note that at a distance s from the starting point, the original cosine μ of the neutron direction will be changed to

$$\mu(s) = \frac{s + r\,\mu}{\sqrt{r^2 + s^2 + 2rs\mu}}. \tag{48}$$

With this, the cosine of the scattering angle at the collision site can be written as

$$\mu_0(\mu(s),\, \mu',\, \varphi') = \sqrt{1 - \mu^2(s)}\,\sqrt{1 - \mu'^2}\cos(\varphi') + \mu(s)\,\mu', \tag{49}$$

where μ' and φ' are the directional cosine and azimuthal angle of the outgoing neutron, also with respect to the position vector of the neutron at $r'(s)$. $(\varphi = 0)$.

Thus one has to perform the integral with the Dirac-delta function in the form

$$\boxed{\delta\left(\sqrt{1 - \mu^2(s)}\,\sqrt{1 - \mu'^2}\cos(\varphi') + \mu(s)\,\mu' - S(E,E')\right)} \tag{50}$$

Elastic scattering term (cont)

The integral of (50) was attempted w.r.t. both μ' and φ', without success.

But following Davison (advice of M.M.R. Williams), it turns out that the integration with respect to the energy, with the given form of $S(E,E')$ in (36) can be performed analytically.

To this end we write

$$\begin{aligned}\delta\left(\mathbf{\Omega}\cdot\mathbf{\Omega}' - S(E,E')\right) &= \delta\left(\mu_0 - \frac{1}{2}\left[(A+1)\sqrt{\frac{E'}{E}} - (A-1)\sqrt{\frac{E}{E'}}\right]\right) = \\ &= \delta\left(\mu_0 - \frac{(A+1)\,E' - (A-1)\,E}{2\sqrt{E\,E'}}\right)\end{aligned} \tag{51}$$

The elastic scattering

Then, with a long algebra, using the expression for the Dirac-delta of a function of a variable, one has

$$\delta\left(\mu_0 - \frac{(A+1)\,E' - (A-1)\,E}{2\sqrt{E\,E'}}\right) = \frac{2E}{(A+1)^2}\,\frac{\left[\mu_0 + \sqrt{\mu_0^2 + A^2 - 1}\right]^2}{\sqrt{\mu_0^2 + A^2 - 1}}\,\delta(E' - E_0') \tag{52}$$

with

$$E_0'(\mu(s), \mu', \varphi') = \frac{E}{(A+1)^2}\left[\mu_0 + \sqrt{\mu_0^2 + A^2 - 1}\right]^2 \tag{53}$$

Hence the scattering function of (37) can be written as

$$\boxed{f_{el}(\mathbf{\Omega}\cdot\mathbf{\Omega}', E \to E') = \frac{1}{4\pi A}\,\frac{\left[\mu_0 + \sqrt{\mu_0^2 + A^2 - 1}\right]^2}{\sqrt{\mu_0^2 + A^2 - 1}}\,\delta(E' - E_0'(\mu(s), \mu', \varphi'))} \tag{54}$$

The generating function

With the above, the final form of the equation for the generating function will look as

$$
\begin{aligned}
& g(z|\, r,\, \mu,\, E) = z\, e^{-\ell(r,\mu)\Sigma(E)} + \\
& \frac{\Sigma_{el}(E)}{4\pi A} \int_0^{\ell(r,\mu)} \mathrm{d}s\, e^{-s\,\Sigma(E)} \times \\
& \int_0^{2\pi} \mathrm{d}\varphi' \int_{-1}^{1} \mathrm{d}\mu' \, \frac{\left[\mu_0 + \sqrt{\mu_0^2 + A^2 - 1}\right]^2}{\sqrt{\mu_0^2 + A^2 - 1}}\, g\Big(z\,|r'(s), \mu',\, E_0'(\mu(s), \mu', \varphi')\Big) \qquad (55) \\
& + \Sigma_{in}(E) \int_0^{\ell(r,\mu)} \mathrm{d}s\, e^{-s\,\Sigma(E)} \int_0^{E} \mathrm{d}E'\, f_{in}(E \to E')\, g(z|\, r'(s),\, E') \\
& + \Sigma_f(E) \int_0^{\ell(r,\mu)} \mathrm{d}s\, e^{-s\,\Sigma(E)}\, q_f \left[\int_0^{E_{max}} \mathrm{d}E'\, \chi(E \to E')\, g(z|\, r'(s), E'),\, E'\right]
\end{aligned}
$$

Factorial moments

First moment (mean number of neutrons $n(r,\,\mu,\,E)$ leaving the item)

$$
\begin{gathered}
n(r,\,\mu,\,E) = e^{-\ell(r,\mu)\Sigma(E)} + \\
\frac{\Sigma_{el}(E)}{4\pi A}\int_0^{\ell(r,\mu)} \mathrm{d}s\, e^{-s\,\Sigma(E)} \times \\
\int_0^{2\pi} \mathrm{d}\varphi' \int_{-1}^{1} \mathrm{d}\mu' \,\frac{\left[\mu_0 + \sqrt{\mu_0^2 + A^2 - 1}\right]^2}{\sqrt{\mu_0^2 + A^2 - 1}}\, n\Big(r'(s), \mu',\, E_0'(\mu(s), \mu', \varphi')\Big) \\
+ \Sigma_{in}(E)\int_0^{\ell(r,\mu)} \mathrm{d}s\, e^{-s\,\Sigma(E)} \int_0^{E} \mathrm{d}E'\, f_{in}(E \to E')\, n(r'(s),\, E') \\
+ \nu_{f,1}(E)\,\Sigma_f(E)\int_0^{\ell(r,\mu)} \mathrm{d}s\, e^{-s\,\Sigma(E)}\, [\int_0^{E_{max}} \mathrm{d}E'\, \chi(E \to E')\, n(r'(s),\, E')
\end{gathered}
\tag{56}
$$

It is indicated that the neutron multiplicities, and hence the factorial moments $\nu_{f,i}(E)$ of the induced fission, are now energy dependent.

Similar equations are obtained for the second and third moments.

Source event induced distributions

One needs the (normalised) energy spectrum $\chi_s(E)$ of the source neutrons.

The generating function $G(z)$ of the number distribution of neutrons leaving the item for a source event:

$$G(z) = \frac{1}{R} \int_V r^2\, q_s \left[\int g(z\,|r,E)\,\chi_s(E)\, \mathrm{d}E \right] \mathrm{d}r \tag{57}$$

Only the scalar single particle induced genreation function appears in this experssion, weighted with the energy spectrum $\chi_s(E)$ of the source neutrons.

Hence, define the spontaneous fission spectrum weighted single particle induced factorial moments as

$$\overline{n}(r) \equiv \int n(r,E)\,\chi_s(E)\,\mathrm{d}E; \qquad \overline{m}(r) \equiv \int m(r,E)\,\chi_s(E)\,\mathrm{d}E$$

and (58)

$$\overline{w}(r) \equiv \int w(r,E)\,\chi_s(E)\,\mathrm{d}E.$$

Source event induced distributions

With these angularly integrated and source spectrum weighted moments, for a spherical item, we obtain expressions ffor the moments N, M and W in perfect analogy with those of the one-speed case, Eqs (23) - (25) as

$$N = \frac{3\,\nu_{s,1}}{R^3}\int_0^R r^2\,\overline{n}(r)\,\mathrm{d}r, \tag{59}$$

$$M = \frac{3}{R^3}\int_0^R r^2\left\{\nu_{s,2}\,\overline{n}^2(r) + \nu_{s,1}\,\overline{m}(r)\right\}\,\mathrm{d}r \tag{60}$$

and

$$W = \frac{3}{R^3}\int_0^R r^2\left\{\nu_{s,3}\,\overline{n}^3(r) + 3\,\nu_{s,2}\,\overline{n}(r)\,\overline{m}(r) + \nu_{s,1}\,\overline{w}(r)\right\}\,\mathrm{d}r. \tag{61}$$

Determination of the mass of the fissile isotope

Traditional point model:

- The multiplicities are explicit functions of the lumped parameters of the item, which can be unfolded analytically
- Out of these lumped parameters, only the spontaneous fission rate $\boldsymbol{F}$ appears explicitly, which only gives the mass of the fertile isotope, e.g. the mass of the ^{240}Pu content, in a mixture of ^{239}Pu and ^{240}Pu.
- The mass of the ^{239}Pu, can only be determined if the isotopic ratio of the two components is known, which often requires the use of methods of destructive assay.

The space- (and energy-) dependent model:

- In the integral equations for the multiplicities, all physical sample parameters (mass of ^{239}Pu and ^{240}Pu) appear explicitly
- The parameters cannot be unfolded analytically. Machine learning can be used (artificial neural networks)
- The formalism has to be extended for a mixture of two (or more) isotopes.

The structure of the ANN

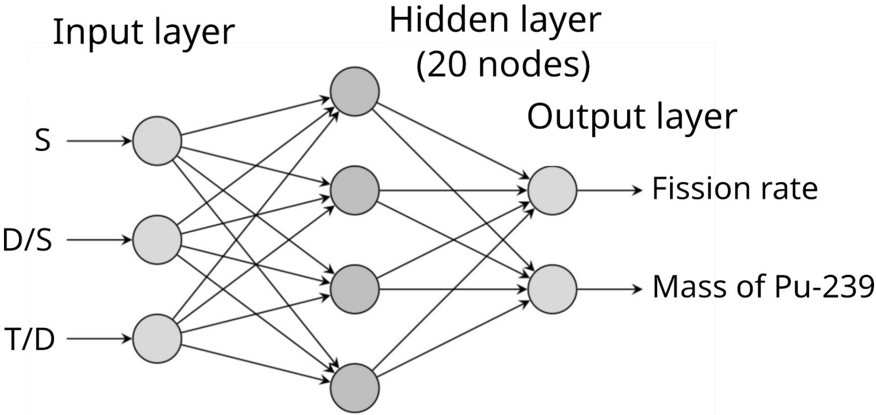
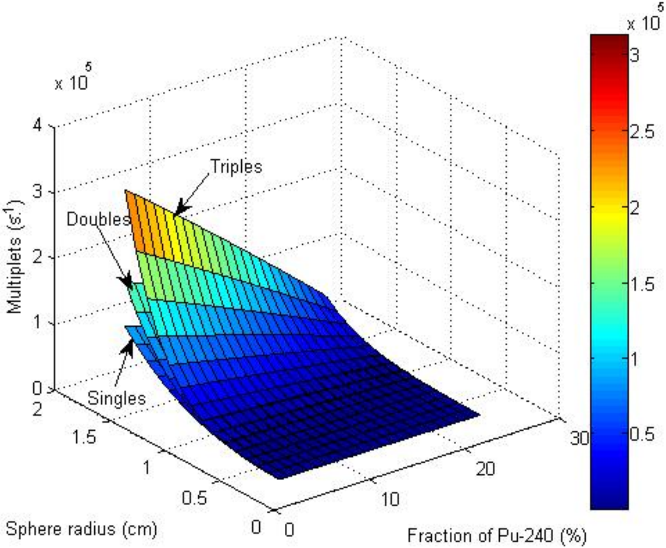


Figure 4: A standard feed-forward ANN used in the unfolding calculations.

Input data for the training of the ANN



Results

Results taken from

Avdic S., Dykin V., Croft S. and Pázsit, I. "Item identification with a space-dependent model of multiplicities and artificial neural networks." *Nucl. Inst. Meth.* **A 1057**, 168800 (2023)

Relative error (%)	^{239}**Pu mass**	**Fission rate**
maximum	0.081	0.0077
minimum	-0.0378	-0.0088
mean	8.14e-05	3.02e-05
standard deviation	0.0068	0.0018

Table 1: Relative errors of the ANN outputs compared to the target values for the input data without noise.

Conclusions

There is a renewed interest and activity with neutrons fluctuations in low power systems in various areas:

- Start-up with a weak source
- Extending the "point model" of multiplicity counting to space-angle-**energy** dependent cases (on-going)
- using continuous signals of fission chambers for safeguards and reactivity determination
- Stochastic modelling of the detection of scintillation detectors with Geant4
- etc...